arXiv:math-ph/0409042v1 18 Sep 2004

Extended Voros product in the coherent states framework

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February 7, 2008

Abstract

Using coherent states of the Weyl-Heisenberg algebra h_N , extended Voros products and Moyal brakets are derived. The covariance of Voros product under canonical transformations is discussed. Star product related to Barut-Girardello coherent states of the Lie algebra $su(1,1)$ is also considered. The star eigenvalue problem of singular harmonic oscillator is investigated.

1 Introduction

Deformation quantization is an idea to quantize classical mechanical systems without using operator theory but by deforming a Poisson algebra on a manifold into a noncommutative algebra. In classical mechanics, observables are smooth functions on phase space, which constitute a Poisson algebra, wile in quantum mechanics, the observables constitute a non-commutative associative algebra. In 1949, Moyal introduced a new bracket for functions on the classical phase space that replaces the Poisson one in the quantization procedure [1]. This bracket is closely related to Weyl's correspondence rule between classical and quantum observables [2]. The new Lie algebra associated with this bracket is a deformation of the Poisson algebra. In recent times , deformation quantization has been explored in several contexts : in the strings theory approach to non-commutative geometry [3], matrix model [4], the non-commutative Yang-Mills theories [5] and non-commutative gauge theories [6]. The apparence of noncommutatity in high energy physics helped to revive the deformation quantization technique which was elucited further in [7]. Another star product due to Grosse and Presnajder [8], using generalized coherent states [9], leads to a quantization scheme analogue to Berezin one [10]. Through this article, we will rely on the concept of coherent states which have important application building a bridge between the classical and the quantum worlds views. In other hand, any set of coherent states satisfy two important properties: continuity and identity to unity. The resolution to unity and non-orthogonality are key ingredients to formulate the Voros product in the coherent states framework (Moyal and Voros products are equivalents) as it has been shown by Stern and al [11](see also [12]). In the same spirit, coherent states of quantum system with nonlinear spectrum has been considered to find new kinds of star products [13] and the $su(2)$ coherent states has been nicely used to introduce a new star product on the fuzzy sphere [14].
Usually the coherent states of the Weyl-Heisenberg algebra h_N are constructed by taking the vacuum as reference state and they are eigenvectors of annihilation operators (standard coherent states) [15]. However, they can be also defined by acting the unitary displacement operator on arbitrary element of the representation space of h_N algebra leading to the so-called generalized coherent states [9]. The first main of this work is to point out to extend the standard Voros star product using these coherent states. In the next section, we first review some basic facts about coherent states of h_N algebra [9] (See also the references quoted in [16-17]), and the associated Voros product. Using coherent states constructed from arbitrary reference state, we generalize the standard Voros product. In the best of our knowledge, this extension has been not discussed previously. As illustration, we consider Landau problem. The third section concern the unitary transformation in the noncommutative space spanned by the variables associated to generators of the algebra h_N. It will be shown that the covariance is violated under arbitrary unitary transformations and the particular case in which the covariance is assured corresponds to the canonical transformations of h_N elements. We discuss also the link between squeezing of coherent states and canonical transformation in the noncommutative space generated by eigenvalues variables associated to generators of Weyl-Heisenberg algebra. The

last section is devoted to star product defined by mean of $su(1,1)$ coherent states. As application of this new star product, we investigate the star analogue of eigenvalue problem of the the so-called singular harmonic oscillator. The quantization of the quantum mechanics of this system is different from Berezin one [10]. Concluding remarks close this paper.

2 Coherent states and star products

In this section, we give a review of standard coherent states [15] and derive the associative Voros star product using these states. We first consider $n = 2N$ canonical a_i^+ (boson creation) and a_i^- (boson destruction) operators, $i = 1, 2, ..., N$. The set of operators $\{a_i^+, a_i^-\}$ and the unity close the Weyl-Heisenberg algebra h_N which describe a bosonic system with N degrees of freedom. The operators a_i^+ and a_i^- act in the Hilbert space $\mathcal{H}_b = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes .... \otimes \mathcal{H}_N$ generated by the states

$|n_1, n_2, ..., n_N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes ... \otimes |n_N\rangle$ where n_i are nonnegative integers. The coherent states for h_N algebra are defined as eigenstates of the operators a_i^- with eigenvalues z_i:

$$|\vec{z}\rangle \equiv |z_1, z_2, ..., z_N\rangle = e^{-\sum_{i=1}^{N} |z_i|^2} \sum_{n_i=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} ... z_N^{n_N}}{\sqrt{n_1! n_2! ... n_N!}} |n_1, n_2, ..., n_N\rangle \tag{1}$$

The states $|\vec{z}\rangle$ can be used, following the method presented in [10], to introduce an associative star product.

To every operator O_1 acting on the Hilbert space $\mathcal{H}_b$

$$O_1 = \sum_{\vec{n},\vec{m}} (O_1)_{\vec{n},\vec{m}} \quad (a_1^+)^{m_1} ... (a_N^+)^{m_N} (a_1^-)^{n_1} ... (a_N^-)^{n_N} \tag{2}$$

where $\vec{m} = (m_1, ..., m_N)$ and $\vec{n} = (n_1, ..., n_N)$, one can associate a function $\mathcal{O}_1(\vec{\bar{z}}, \vec{z})$ according

$$\mathcal{O}_1(\vec{\bar{z}}, \vec{z}) = \langle \vec{z} | O_1 | \vec{z} \rangle = \sum_{\vec{n},\vec{m}} (O_1)_{\vec{n},\vec{m}} \quad (\bar{z}_1)^{m_1} ... (\bar{z}_N)^{m_N} (z_1)^{n_1} ... (z_N)^{n_N} \tag{3}$$

An associative star product of two functions $\mathcal{O}_1(\vec{\bar{z}}, \vec{z})$ and $\mathcal{O}_2(\vec{\bar{z}}, \vec{z})$ is then defined by

$$\mathcal{O}_1(\vec{\bar{z}}, \vec{z}) \star \mathcal{O}_2(\vec{\bar{z}}, \vec{z}) = \langle \vec{z} | O_1 O_2 | \vec{z} \rangle \tag{4}$$

The associativity of the this star product originates from associativity of the Weyl-Heisenberg algebra h_N. Since the coherent states $|\vec{z}\rangle$ are eigenstates of the operators a_i^- with the eigenvalues z_i, it is easy to verify that

$$\bar{z}_i \star z_j = \bar{z}_i z_j \tag{5}$$

$$\bar{z}_i \star \bar{z}_j = \bar{z}_i \bar{z}_j = \bar{z}_j \star \bar{z}_i \tag{6}$$

$$z_i \star z_j = z_i z_j = z_j \star z_i \tag{7}$$

$$z_j \star \bar{z}_i = \delta_{ij} + z_j \bar{z}_i \tag{8}$$

The Moyal Brackets are given by

$$\{z_j, \bar{z}_i\}_\star = z_j \star \bar{z}_i - \bar{z}_i \star z_j = \delta_{ij} \tag{9}$$

reflecting that the structure relations of h_N algebra are preserved in the star quantization. Note that the relations (5-8) constitute the minimal set of relations to compute the $\star$-product of any two arbitrary functions $\mathcal{O}_1(\vec{\bar{z}}, \vec{z})$ and $\mathcal{O}_2(\vec{\bar{z}}, \vec{z})$. Note also that the coherent states $|\vec{z}\rangle$ correspond to unitary action on the ground state $|0, 0, ..., 0\rangle$ of the multimode bosonic system under consideration. As mentionned in the introduction, general sets of coherent states can be constructed by acting the unitary operator

$$D(\vec{z}) = e^{\sum_{i=1}^{N}(z_i a_i^+ - \bar{z}_i a_i^-)} \tag{10}$$

on arbitrary reference state $|k_1, k_2, ..., k_N\rangle$ and can used to define a new star products, extending the Voros one. The action of the operator $D(\vec{z})$ on the reference state $|k_1, k_2, ..., k_N\rangle \equiv |\vec{k}\rangle$:

$$|\vec{z}, \vec{k}\rangle = \otimes_{i=1}^{N} D(z_i)|k_i\rangle \equiv \otimes_{i=1}^{N}|z_i, k_i\rangle \tag{11}$$

where the vectors $|z_i, k_i\rangle$ are expressed as follows

$$|z_i, k_i\rangle = e^{-\frac{1}{2}z_i\bar{z}_i}\Big(\sum_{l_i\leq k_i}\sqrt{\frac{l_i!}{k_i!}}(-\bar{z}_i)^{k_i-l_i}L_{l_i}^{k_i-l_i}(|z_i|^2)|l_i\rangle + \sum_{l_i\geq k_i}\sqrt{\frac{k_i!}{l_i!}}(z_i)^{l_i-k_i}L_{k_i}^{l_i-k_i}(|z_i|^2)|l_i\rangle\Big) \tag{12}$$

in term of the Laguerre polynomials L_n^α.
The states (11) resolve the unity operator for any $\vec{k} = (k_1, k_2, ..., k_N) \in \mathbf{N}^N$ in respect to the measure $\frac{d^2\vec{z}}{\pi^N}$.
The associative product of two functions $\mathcal{O}_1(\vec{\bar{z}}, \vec{z})$ and $\mathcal{O}_2(\vec{\bar{z}}, \vec{z})$ is defined, in this case, by

$$\mathcal{O}_1(\vec{\bar{z}}, \vec{z}) \star_k \mathcal{O}_2(\vec{\bar{z}}, \vec{z}) = \int d^2\vec{z'}\langle\vec{z}, \vec{k}|O_1|\vec{z'}, \vec{k}\rangle\langle\vec{z'}, \vec{k}|O_2|\vec{z}, \vec{k}\rangle \tag{13}$$

Noticing that

$$\prod_{i=1}^{N} e^{-z_i\frac{\partial}{\partial z'_i}} e^{z'_i\frac{\partial}{\partial z_i}} \mathcal{O}_1(\vec{\bar{z}}, \vec{z}) = \frac{\langle\vec{z}, \vec{k}|O_1|\vec{z'}, \vec{k}\rangle}{\langle\vec{z}, \vec{k}|\vec{z'}, \vec{k}\rangle} \tag{14}$$

and

$$\prod_{i=1}^{N} e^{-\bar{z}_i\frac{\partial}{\partial \bar{z}'_i}} e^{\bar{z}'_i\frac{\partial}{\partial \bar{z}_i}} \mathcal{O}_2(\vec{\bar{z}}, \vec{z}) = \frac{\langle\vec{z'}, \vec{k}|O_2|\vec{z}, \vec{k}\rangle}{\langle\vec{z'}, \vec{k}|\vec{z}, \vec{k}\rangle}, \tag{15}$$

one can write the $\star_k$-product as

$$\star_k = \int d^2\vec{z} : e^{\sum_{i=1}^{N}\frac{\partial}{\partial z_i}(z'_i - z_i)} : |\langle\vec{z}, \vec{k}|\vec{z'}, \vec{k}\rangle|^2 : e^{\sum_{i=1}^{N}(\bar{z}'_i - \bar{z}_i)\frac{\partial}{\partial \bar{z}_i}} : \tag{16}$$

where the symbol $: \; :$ stand for an ordered exponential. Clearly the star product is completely determined once one known the overlapping, between two coherent states (11), which is given by

$$|\langle\vec{z}, \vec{k}|\vec{z'}, \vec{k}\rangle|^2 = \prod_{i=1}^{N} e^{-|z'_i - z_i|^2}(L_{k_i}^0(|z'_i - z_i|^2))^2 \tag{17}$$

Substituting (17) in (16) and with a simple change of variables, one get

$$\star_k = \prod_{i=1}^{N} \star_{k_i} \tag{18}$$

where

$$\star_{k_i} = \sum_{p=0}^{\infty} I_{k_i p} \frac{\overleftarrow{\partial^p}}{\partial z_i^p} \frac{\overrightarrow{\partial^p}}{\partial \bar{z}_i^p} \tag{19}$$

The coefficients $I_{k_i p}$ occuring in the last equation are given by

$$I_{k_i p} = \sum_{j,j'=0}^{k_i} \frac{(p+j+j')!}{(p!)^2} (-)^{j+j'} \binom{k_i}{k_i - j} \binom{k_i}{k_i - j'} \tag{20}$$

In the particular case $k_1 = k_2 = ... = k_N = 0$, we have $I_{0p} = \frac{1}{p!}$ and we recover the Voros star product

$$\star = \star_0 = e^{\sum_{i=1}^{N} \frac{\overleftarrow{\partial^p}}{\partial z_i^p} \frac{\overrightarrow{\partial^p}}{\partial \bar{z}_i^p}}. \tag{21}$$

So, it becomes clear that the coherent states (11) leads to a new star product generalizing the Voros one. To compute the $\star_k$-product, of two arbitrary functions, the desired relations are

$$\bar{z}_i \star_k z_j = I_{kN} \bar{z}_i z_j \tag{22}$$

$$\bar{z}_i \star_k \bar{z}_j = I_{kN} \bar{z}_i \bar{z}_j = \bar{z}_j \star_k \bar{z}_i \tag{23}$$

$$z_i \star_k z_j = I_{kN} z_i z_j = z_j \star_k z_i \tag{24}$$

$$z_j \star_k \bar{z}_i = I_{kN} [\delta_{ij} \frac{I_{k_i 1}}{I_{k_i 0}} + z_j \bar{z}_i] \tag{25}$$

where the c-number I_{kN} is defined by $I_{kN} = \prod_{i=1}^{N} I_{k_i 0}$. In this case, the extended Moyal brackets are given by

$$\{z_j, \bar{z}_i\}_{\star_k} = z_j \star_k \bar{z}_i - \bar{z}_i \star_k z_j = I_{kN} \frac{I_{k_i 1}}{I_{k_i 0}} \delta_{ij} \tag{26}$$

Note that for $k_1 = k_2 = ... = k_N = 0$, the relations (26) reduces to ones given by (9) and we recover the usual Moyal brackets.
The physical motivation of the extension of the Voros product can be found in the well known Landau problem. More precisely, the higher landau levels quantum mechanics can be equivalently formulated in a noncommuative setting involving the extended Voros products and to each Landau level k a class of Voros product $\star_k$ can be associated. Before this simple illustration, recall that the Landau spectrum is made of degenerate Landau levels $E_k = 2k + 1$ ($k \geq 0$) with kth Landau level eigenstates labelled by the radial/orbital quantum numbers k , $l \geq 0$ (analytic) and $k + l$, $-k \leq l \leq 0$ (anti-analytic). There are, in a given Landau level, an infinite number of analytic eigenstates

$$\Phi_{k,l}(z, \bar{z}) = e^{-\frac{1}{2}|z|^2} z^l L_k^l(|z|^2) \qquad l \geq 0 \tag{27}$$

and a finite number of anti-analytical eigenstates

$$\Phi_{k,k+l}(z,\bar{z}) = e^{-\frac{1}{2}|z|^2}\bar{z}^{-l}L_{k+l}^{-l}(|z|^2) \qquad -k \leq l < 0 \tag{28}$$

It is interesting to note that the analytic and anti-analytic functions of the kth Landau level can be written also as

$$\Phi_{k,l}(z,\bar{z}) = (-)^l\sqrt{\frac{(k+l)!}{k!}}\langle k+l|z,k\rangle \qquad l \geq 0 \tag{29}$$

and

$$\Phi_{k,k+l}(z,\bar{z}) = \sqrt{\frac{k!}{(k+l)!}}\langle k+l|z,k\rangle \qquad -k \leq l < 0 \tag{30}$$

It is clear that the functions $\Phi_{k,l}(z,\bar{z})$ (resp. $\Phi_{k,k+l}(z,\bar{z})$) corresponds to analytic (resp. anti-analytic) representations of the coherent state $|z,k\rangle$ (Equation (11) for one bosonic degree of freedom) constructed from the fiducial vector $|k\rangle$ (eigenstate of the Landau Hamiltonian). In view of the above considerations on the extension of the Voros product, it results that in a given Landau level k, the noncommutativity can be introduced through $\star_k$ product between the analytic representations of degenerate states. Of course, this question requires more analysis which will be considered in another work.

3 Canonical covariance

Let us start by examining the covariance in the noncommutative space, endowed with Voros product, under unitary transformations. For simplicity reasons, we restrict ourself to one bosonic degree of freedom. Let O be an operator and $U = e^{\Lambda}$ ($\Lambda^+ + \Lambda = 0$) a one or multi-parameter unitary transformation of the operator O in the operator space. First, we will show that the unitary transformations in the operator space has unique representation in the noncommutative space generated by the variables z and $\bar{z}$. In other words, we will show that the function $\mathcal{O}'(z,\bar{z})$ associated to the operator $O' = U^+OU$ is given by

$$\mathcal{O}'(z,\bar{z}) = e^{-\mathcal{D}_\lambda}\mathcal{O}(z,\bar{z}) \tag{31}$$

where the function $\mathcal{O}(z,\bar{z}) = \langle z|O|z\rangle$ and $\mathcal{D}_\lambda$ acts as

$$\mathcal{D}_\lambda\mathcal{O}(z,\bar{z}) = \lambda(z,\bar{z}) \star \mathcal{O}(z,\bar{z}) - \mathcal{O}(z,\bar{z}) \star \lambda(z,\bar{z}) \tag{32}$$

with $\lambda(z,\bar{z}) = \langle z|\Lambda|z\rangle$.
As we mentioned in the previous section, any operator O in the operator algebra can be expanded in terms of creation and annihilation operators a^+ and a^-

$$O = \sum_{m,n} O_{m,n}(a^+)^m(a^-)^n \tag{33}$$

The proof of the relation (31) is immediate. Indeed, expanding the operator O' as

$$O' = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Big[\Lambda[\Lambda, ..., [\Lambda, O]...]\Big] \tag{34}$$

and using the star calculus in the coherent states scheme, it is easily seen that the function $\mathcal{O}'(z,\bar{z})$ is given by

$$\mathcal{O}'(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Big\{\lambda(z,\bar{z})\{\lambda(z,\bar{z}), ..., \{\lambda(z,\bar{z}), \mathcal{O}(z,\bar{z})\}_\star ...\}_\star\Big\}_\star \tag{35}$$

A result which can be also written in the compact form as in (31) where $\mathcal{D}_\lambda$ is defined through Eq(32). Therefore, each unitary transformation in the operator space has unique representation in the noncommutative space and there a correspondance between unitary transformations in the operator space and ones given by (31). Note also that we have the following identities

$$e_\star^{-\lambda} \star e_\star^{\lambda} = e_\star^{\lambda} \star e_\star^{-\lambda} = 1 \tag{36}$$

where the star exponential is defined as

$$e_\star^{X} = \sum_{k=0}^{\infty} \frac{1}{k!}(X)^{\star k} = \sum_{k=0}^{\infty} \frac{1}{k!} X \star X \star ... \star X \tag{37}$$

Remark that if we consider two transformations acting on the operator space $O_1 = e^{\Lambda_1}$ and $O_2 = e^{\Lambda_2}$, one has

$$[\mathcal{D}_{\lambda_1}, \mathcal{D}_{\lambda_2}] = \mathcal{D}_{\{\lambda_1,\lambda_2\}_\star} \tag{38}$$

where the functions λ_1 and λ_2 are ones associated with anti-hermitians operators Λ_1 and Λ_2, respectively.
Using the harmonic coherent states, the function associated to the operator O (Eq.(33)) is

$$\mathcal{O}(z,\bar{z}) = \sum_{mn} O_{mn} \bar{z}^m z^n, \tag{39}$$

and one corresponding to the operator $O' = U^+ O U$ is given by

$$\mathcal{O}'(z,\bar{z}) = \langle z|e^{-\Lambda} O e^{+\Lambda}|z\rangle \tag{40}$$

Thanks to the definition (4), the function $\mathcal{O}'(z,\bar{z})$ can be written as

$$\mathcal{O}'(z,\bar{z}) = \langle z|e^{-\Lambda}|z\rangle \star \langle z|O|z\rangle \star \langle z|e^{+\Lambda}|z\rangle \tag{41}$$

and combining the definitions (4), (32) and (41), one has

$$\mathcal{O}'(z,\bar{z}) = e_\star^{-\lambda(z,\bar{z})} \star \mathcal{O}(z,\bar{z}) \star e_\star^{+\lambda(z,\bar{z})} \tag{42}$$

Alternatively, It can be expressed as follows

$$\mathcal{O}'(z,\bar{z}) = \sum_{mn} O_{mn} \langle z|e^{-\Lambda}(a^+)^m e^{+\Lambda}|z\rangle \star \langle z|e^{-\Lambda}(a^-)^n e^{+\Lambda}|z\rangle \tag{43}$$

$$\mathcal{O}'(z,\bar{z}) = \sum_{mn} O_{mn} \bar{Z}^{\star m} \star Z^{\star n} \tag{44}$$

where the new variables Z and $\bar{Z}$ are defined by

$$Z = e^{-\mathcal{D}_\lambda} z \qquad\qquad \bar{Z} = e^{-\mathcal{D}_\lambda} \bar{z} \tag{45}$$

It is evident that

$$\mathcal{O}'(z,\bar{z}) \neq \mathcal{O}(Z,\bar{Z}) \tag{46}$$

in general. Thus, the covariance under arbitrary unitary transformation is violated. The considerations presented here can be extended to multi-dimensional noncommutative space in a straightfoward way.
One exceptional case in which the equality in (46) holds can be identified corresponds to the group of linear canonical transformations. To examine this exceptional case, let us first recall the canonical transformations for a bosonic system with a finite number of degrees of freedom. We shall consider the transformations preserving the commutations relations of h_N algebra

$$A_i^- = T^+ a_i^- T. \tag{47}$$

The operator T is given by

$$T = \exp\left(\frac{\xi_{ij}}{2} a_i^+ a_j^+ - \frac{\bar{\xi}_{ij}}{2} a_i^- a_j^-\right) \tag{48}$$

where ξ_{ij} are elements of complex symmetrical matrix that will denoted by Ξ. In (48), summation over repeated indices is underlying. The operator T realize a representation of the group $Sp(2N,\mathbf{R})$. By a direct computation, one can write the transformation (47) as follows

$$A^{\mp} = \cosh\sqrt{\Xi^+\Xi}\, a^{\mp} + \frac{\Xi}{\sqrt{\Xi^+\Xi}} \sinh\sqrt{\Xi^+\Xi}\, a^{\pm} \tag{49}$$

in a compact form in term of the complex symmetrical matrix Ξ.
In other hand, the function corresponding to the operator $\Lambda = \frac{\xi_{ij}}{2} a_i^+ a_j^+ - \frac{\bar{\xi}_{ij}}{2} a_i^- a_j^-$ (Eq.(48)) takes the following form

$$\lambda(\vec{z},\vec{\bar{z}}) = \frac{1}{2}\xi_{ij}\bar{z}_i\bar{z}_j - \frac{1}{2}\bar{\xi}_{ij} z_i z_j \tag{50}$$

Using the latter expression of the function $\lambda(\vec{z},\vec{\bar{z}})$, one can evaluate the action of the operator $\mathcal{D}_\lambda$ on canonical variables z_k and $\bar{z}_k$. We obtain

$$\mathcal{D}_\lambda z_k = -\xi_{ki}\bar{z}_i \qquad \mathcal{D}_\lambda \bar{z}_k = -\xi_{ki} z_i \tag{51}$$

which can be written also in compact form as

$$\mathcal{D}_\lambda z = -\Xi\bar{z} \qquad \mathcal{D}_\lambda \bar{z} = -\Xi^+ z \tag{52}$$

From the later results, one show

$$e^{-\mathcal{D}_\lambda} z = \cosh \sqrt{\Xi^+\Xi} z + \frac{\Xi}{\sqrt{\Xi^+\Xi}} \sinh \sqrt{\Xi^+\Xi} \bar{z} = Z \tag{53}$$

and

$$e^{-\mathcal{D}_\lambda} \bar{z} = \cosh \sqrt{\Xi^+\Xi} \bar{z} + \frac{\Xi^+}{\sqrt{\Xi^+\Xi}} \sinh \sqrt{\Xi^+\Xi} z = \bar{Z} \tag{54}$$

to be compared with (49). Following the prescription presented in the section 2, the coherent states associated to the algebra generated by the creation and annihilation operators $\{A_i^+, A_i^-; i = 1, 2, ..., N\}$ leads to a star product of Voros type

$$\tilde{\star} = \exp\,(\sum_{i=1}^{N} \overleftarrow{\frac{\partial}{\partial Z_i}} \overrightarrow{\frac{\partial}{\partial \bar{Z}_i}}), \tag{55}$$

and we have the following star commutation relations

$$\bar{Z}_i \tilde{\star} Z_j - Z_j \tilde{\star} \bar{Z}_i = \delta_{ij} \qquad \bar{Z}_i \tilde{\star} \bar{Z}_j = \bar{Z}_j \tilde{\star} \bar{Z}_i \qquad Z_i \tilde{\star} Z_j = Z_j \tilde{\star} Z_i \tag{56}$$

The new variables Z_i corresponds to analytic representation of the operators A_i^-. Because the transformations are canonical, the $\star$ and $\tilde{\star}$-products are identical and we obtain

$$\mathcal{O}'(z, \bar{z}) = \mathcal{O}(Z, \bar{Z}). \tag{57}$$

To close this section, some remarks are in order. First, recall that in any deformation quantization scheme the classical covariance must be assured. In our case, this covariance is guaranted when the transformations in the operator space are of the form given by (47). If one consider one bosonic degree of freedom, the operator (48) is nothing but the so-called squeezing operator. His action on the coherent states generates squeezed states which minimize also the the Heisenberg uncertainty relation but with different variances of the creation and annihilation operators. Then, it seems that the correspondence operator-function using coherent states provides a deeper insight into the properties of quantum state that can not be seen from the analytical functions. There are a parrallelism between canonical covariance in the noncommutative space and the transformation (47) which transforms a coherent state in squeezed one in a way preserving the minimal value of the Heisenberg inequality. It will be amusing to study this parrallelism in the light of the recent results related to quantum correlations and the extended phase space discussed in [18].

4 $su(1,1)$ star product

In 1971, Barut and Girardello [19] defined the coherent states of $su(1,1)$ algebra as eigenstates of the lowering operator K_-. They are given by

$$|z\rangle \;=\; \frac{|z|^{k-1/2}}{(I_{2k-1}(2|z|))^{1/2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(n+1)\Gamma(n+2k)}} |n, k\rangle \tag{58}$$

where $I_{2k-1}(2|z|)$ is the modified Bessel function and k stand for discrete series of $su(1,1)$ unitary irreducible representations. The Hilbert space is spanned by the complete orthonormal basis $|n,k\rangle$. Due to the completion property , any state $|f\rangle$, of the Hilbert space, can be represented by an entire function. In particular, the analytical representations of the vectors $|n,k\rangle$ are

$$\mathcal{F}_n^k(z) = \frac{z^n}{\sqrt{\Gamma(n+1)\Gamma(n+2k)}} \tag{59}$$

The $su(1,1)$ elements K_+, K_- and K_3 are realized, in this representation, by

$$K_+ = z \qquad K_- = z\frac{d^2}{dz^2} + 2k\frac{d}{dz} \qquad K_3 = z\frac{d}{dz} + k \tag{60}$$

Similarly to the Weyl-Heisenberg algebra, to every operator A of $su(1,1)$ algebra acting on the the representation space spanned by the vectors $|n,k\rangle$, one can associate on the complex plane a function $\mathcal{A}(z,\bar{z})$ as

$$\mathcal{A}(z,\bar{z}) = \langle z|A|z\rangle \tag{61}$$

In this respect, to element K_- (resp. K_+) correspond the analytic function $z \to z$ (resp. anti-analytic function $\bar{z} \to \bar{z}$). The star product, in the $su(1,1)$ case, is defined in the same manner that one given by (4). The associativity of this star product is inherited, here again, from the associativity of the usual product of the $su(1,1)$ algebra. It is simple to verify that the star product of two analytic or two anti-analytic functions reduces the usual commutative product of functions. One can also verify that

$$\bar{z} \star z = \bar{z}z \qquad\qquad z \star \bar{z} = z\bar{z} - \Theta_k(z,\bar{z}) \tag{62}$$

where the function $\Theta_k(z,\bar{z})$ is given by

$$\Theta_k(z,\bar{z}) = 2k + \frac{|z|^2}{k}\frac{{}_0F_1(2k+1,|z|^2)}{{}_0F_1(2k,|z|^2)} \tag{63}$$

In (63), ${}_0F_1$ is the well known hypergeometric function. The star product between variables z, $\bar{z}$ and the function $\Theta_k(z,\bar{z})$ are given by

$$\bar{z} \star \Theta_k(z,\bar{z}) = \Theta_k(z,\bar{z}) + 2k\bar{z} - 2k \tag{64}$$

$$\Theta_k(z,\bar{z}) \star \bar{z} = \Theta_k(z,\bar{z}) + (2k+2)\bar{z} - 2k \tag{65}$$

$$\Theta_k(z,\bar{z}) \star z = \Theta_k(z,\bar{z}) + 2kz - 2k \tag{66}$$

$$z \star \Theta_k(z,\bar{z}) = \Theta_k(z,\bar{z}) + (2k+2)\bar{z} - 2k \tag{67}$$

which are useful to compute the star product between any two arbitrary functions of z and $\bar{z}$. The Moyal brackets, in the $su(1,1)$ case, are given by

$$\{z,\bar{z}\}_\star = \Theta_k(z,\bar{z}) \qquad \{z,\Theta_k(z,\bar{z})\}_\star = 2z \qquad \{\bar{z},\Theta_k(z,\bar{z})\}_\star = -2\bar{z} \tag{68}$$

traducing the fact that the $su(1,1)$ structure relations are preserved in the star language. It is well established that there is a contraction procedure reducing the $su(1,1)$ algebra to Weyl-Heisenberg one [19]. Following this way, we will show that the star commutation relations (68) can be contracted to ones corresponding to harmonic oscillator. In this order, we set

$$z' = \sqrt{q}z \qquad \bar{z}' = \sqrt{q}\bar{z} \qquad \Theta'_k(z,\bar{z}) = q\Theta_k(z,\bar{z}) \tag{69}$$

with $q > 0$. The relations (68) becomes

$$\{z',\bar{z}'\}_\star = \Theta'_k(\frac{z'}{\sqrt{q}},\frac{\bar{z}'}{\sqrt{q}}) \qquad \{z',\Theta'_k(\frac{z'}{\sqrt{q}},\frac{\bar{z}'}{\sqrt{q}})\}_\star = 2qz' \qquad \{\bar{z}',\Theta'_k(\frac{z'}{\sqrt{q}},\frac{\bar{z}'}{\sqrt{q}})\}_\star = -2q\bar{z}' \tag{70}$$

In the limit $k \to \infty$ and $q \to 0$ but $2qk \to 1$, by substituting (69) in (63), one get

$$\Theta'_k(\frac{z'}{\sqrt{q}},\frac{\bar{z}'}{\sqrt{q}}) \to 1 \tag{71}$$

and the star structure relations coincides with noncommutative ones associated to harmonic oscillator.
To give an application of the $su(1,1)$ star product, we consider the singular harmonic oscillator. Before going on, recall that the star analogue of eigenvalue problem for n-dimensional noncommutative harmonic oscillator has been recently considered in [20].
The one dimensional harmonic oscillator still integrable if we add a x^{-2} potential and we have the singular harmonic oscillator

$$H_{cal} = a^+a^- + \frac{1}{2} + \frac{\eta^2}{x^2} \tag{72}$$

where $a^\pm = \frac{1}{\sqrt{2}}(x \pm \frac{d}{dx})$ are the usual creation and annihilation operators. This one dimensional system is called also "isotonic oscillator" [21] or two-particles Calogero interaction [22]. The normalized eigenfunctions of the Hamiltonian H_{cal} are

$$\Psi_n(x) = (-)^n\sqrt{\frac{2n!}{\Gamma(n+e_0)}}L_n^{e_0-1}(x^2)\exp(-\frac{x^2}{2}) \tag{73}$$

where $\alpha = \frac{1}{2} + \sqrt{(\frac{1}{4}+2\eta^2)}$. The eigenvalues are given by:

$$H_{cal}\Psi_n(x) = e_n\Psi_n(x) \tag{74}$$

where $e_n = (2n+e_0)$ and $e_0 = \alpha + 1/2$. The waves functions $\Psi_n(x)$ form a basis in the Hilbert space $\mathcal{H}$ of square integrable functions on the half axis $0 < x < \infty$. The raising and lowering operators are defined by

$$A^\pm = \frac{1}{2}((a^\pm)^2 - \frac{\eta^2}{x^2}), \tag{75}$$

and act on the eigenstates $|\Psi_n\rangle$ as follows

$$A^+|\Psi_n\rangle = \sqrt{(n+1)(n+e_0)}|\Psi_{n+1}\rangle \tag{76}$$

and

$$A^-|\Psi_n\rangle = \sqrt{n(n+e_0-1)}|\Psi_{n-1}\rangle. \tag{77}$$

The operaors A^+ , A^- and H_{cal} close the $su(1,1)$ algebra. Thus, the Barut-Girardello coherent states of the singular harmonic oscillator are obtained from the expression (58) by simply setting $2k \equiv e_0 + 1$ and $|n,k\rangle \equiv |\Psi_n\rangle$. Every operator A, acting on the Hilbert space of the quantum system under consideration, can be expanded in the operator basis $\{P_{mn} = |\Psi_m\rangle\langle\Psi_n|\}$ and to each element P_{mn} we associate the function

$$\mathcal{P}_{m,n}(z,\bar{z}) = \langle z|\Psi_m\rangle\langle\Psi_n|z\rangle = \frac{|z|^{e_0}}{I_{e_0}(2|z|)}\mathcal{F}_m^{(e_0+1)/2}(\bar{z})\mathcal{F}_n^{(e_0+1)/2}(z) \tag{78}$$

satisfying the completion relation $\sum_{m=0}^{\infty}\mathcal{P}_{m,n}(z,\bar{z}) = 1$ and the orthogonality property

$$\mathcal{P}_{m,n}(z,\bar{z}) \star \mathcal{P}_{m',n'}(z,\bar{z}) = \delta_{m',n}\mathcal{P}_{m,n'}(z,\bar{z}) \tag{79}$$

It is easy to prove that the $\mathcal{P}_{0,0}(z,\bar{z})$ provides a star vacuum

$$z \star \mathcal{P}_{0,0}(z,\bar{z}) = \mathcal{P}_{0,0}(z,\bar{z}) \star \bar{z} = 0. \tag{80}$$

We have also

$$z \star \mathcal{P}_{m,n}(z,\bar{z}) = \sqrt{m(m+e_0-1)}\mathcal{P}_{m-1,n}(z,\bar{z}) \tag{81}$$

$$\bar{z} \star \mathcal{P}_{m,n}(z,\bar{z}) = \sqrt{(m+1)(m+e_0)}\mathcal{P}_{m+1,n}(z,\bar{z}) \tag{82}$$

$$\mathcal{P}_{m,n}(z,\bar{z}) \star z = \sqrt{(n+1)(n+e_0)}\mathcal{P}_{m,n+1}(z,\bar{z}) \tag{83}$$

$$\mathcal{P}_{m,n}(z,\bar{z}) \star \bar{z} = \sqrt{n(n+e_0-1)}\mathcal{P}_{m,n-1}(z,\bar{z}) \tag{84}$$

The function $\mathcal{H}_{cla}$ associated to the Hamiltonian H_{cal} is given

$$\mathcal{H}_{cla} = \langle z|H|z\rangle = \Theta_{\frac{e_0+1}{2}}(z,\bar{z}) \tag{85}$$

where the function Θ is defined by Eq(63) and one show that

$$\mathcal{H}_{cla} \star \mathcal{P}_{m,n}(z,\bar{z}) = e_m\mathcal{P}_{m,n}(z,\bar{z}) \tag{86}$$

$$\mathcal{P}_{m,n}(z,\bar{z}) \star \mathcal{H}_{cla} = e_n\mathcal{P}_{m,n}(z,\bar{z}) \tag{87}$$

The functions $\mathcal{P}_{n,n}(z,\bar{z})$ are the star eigenstates of $\mathcal{H}_{cla}$ satisfying $\mathcal{H}_{cla} \star \mathcal{P}_{n,n}(z,\bar{z}) = \mathcal{P}_{n,n}(z,\bar{z}) \star \mathcal{H}_{cla}$ and they are real. Finally, remark that the star quantization of singular harmonic oscillator, presented here, is different from the Berezin one discussed in [8,10]. An interesting question concerns a better understanding of the realtionship between the Voros star product using Barut-Girardello states and the noncommutative star product defined in [8] by mean of group-theoretic coherent states. Such relation can be established because the analytical representations of these two sets of coherent states are related through Laplace transforms [23]. Remark also that the deformation quantization discussed in this section can be applied to other quantum mechanical systems (Morse potential, for example).

5 Concluding remarks

To conclude, let us summarize the main points developed in this work. We have investigated an extension of the standard Voros product using coherent states defined from an arbitrary reference state. Conventional star product is recovered when the reference state coincides with the vacuum. We have shown that the covariance of star calculus can not be guaranteed under all unitary transformations except the canonical ones. Another result obtained in this work concerns the star product derived from the Barut-Girardello coherent states of the $su(1,1)$ algebra. As application, we considered the star analogue of the singular harmonic oscillator. It is interesting to use the extended Voros product in the context of deformation quantization of geometric quantum mechanics purposed in [24] and in the scheme of tomographic representation [25] . It is also desirable to extend the formalism presented here to fermionic and supersymmetric systems as well as other Lie algebras like, for instance, $su(p,q)$. We postpone a full study of such questions to future works.

Acknowledgements: The author would like to thank the Condensed matter Section of the Abdus Salam-ICTP for hospitality. He is also grateful to the referees for theirs comments and suggestions.

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